Analysis of the shape of a sheet of paper when two opposite edges are joined

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The profiles of the edge of rectangular sheets of paper folded so as to join opposite edges were measured and compared with the profiles calculated with the theory of elasticity. I show how the calculation can be accomplished by using the aspect ratio of the loops as the only input parameter.
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I. INTRODUCTION

The profile of the edge of a rectangular sheet of paper folded so that two opposite edges are joined is qualitatively similar to the profile of a teardrop (see Fig. 1). In this paper I will describe how I measured the shape of the profile of the paper and how I calculated the shape using the theory of elasticity.

Paper can be considered as an elastic and inextensible material, and the equations of equilibrium for a sheet of paper when it is folded in only one direction are equivalent to those of a thin inextensible rod bent in a plane. In addition to the interest of applying a theory to a practical example, a sheet of paper folded in this way is equivalent to the problem of a thin elastic rod, which is a problem of historical importance in the development of the theory of elasticity.[1] The shape of the folded paper can be calculated without using the values of the elastic parameters or the magnitude of the applied forces. Teachers can consider starting the study of the deformation of thin elastic sheets (such as rubber, metal, paper, and leather) with this example. The same theory has been applied in nanotechnology, where the cross section of a nanotube created by folding a layer on a substrate can be determined.[2,3] The theory is often applied to textiles to obtain elastic properties from an analysis of the form materials take upon bending.[4] We will comment on these cases in Sec. V.

The equations that describe the profile of the sheet of paper are briefly presented in Sec. II. The solutions of these well-known equations can be written in terms of elliptic integrals. Because we will not measure any physical parameter other than various lengths, these solutions will not help us predict the profile. I will show why we cannot directly use the analytical solution. The measured quantities will be only the height and width of the loop. Its ratio, the aspect ratio of the loop, will be used to calculate the profile, which we will compare with the actual profile of a sheet of paper. A scanner and a digital camera was used to obtain the profile of a sheet of paper. Good agreement was observed.

II. THE THEORY

The development of the mathematical theory of elasticity had a milestone in 1744 when Euler obtained the differential equations describing the curvature of a thin elastic and inextensible rod bent in a plane.[1] This problem, known as the problem of the elastic line or *elastica*, had been studied decades ago by James Bernouilli. It was his nephew Daniel Bernouilli who suggested to Euler to look for the differential equation by minimizing the integral of the square of the curvature taken along the rod. Euler used this method to find the equations of the *elastica*.

The shape of a thin rod depends on the forces and moments applied to it. The equations to calculate its shape can also be obtained from the equations of mechanical equilibrium plus the Bernouilli-Euler assumption that the moment at any section within the rod is proportional to the local curvature. When the rod is bent in a plane and all the forces acting on the rod are applied only at isolated points, this method yields[5]

$$EI\frac{d^2\phi(s)}{ds^2} = F_x \sin \phi(s) - F_y \cos \phi(s), \qquad (1)$$

where E is Young's modulus, I is the moment of inertia of the cross section of the rod, s is the arclength along the rod, $\phi(s)$ is the angle between the x axis and the tangent to the rod at s (see Fig. 2), and F_x and F_y are the components of the resultant internal stress on the cross section of the rod. The solution of Eq. (1) with the appropriate boundary conditions provides the shape of a thin rod or the profile of the edge of a rectangular plate when it remains unfolded in one direction.

Static equilibrium implies that F_x and F_y are constant between every two points where the external concentrated forces (forces applied only at isolated points) could be applied. If the external forces act only at the ends of a rod or the edges of a plate, the stress is constant throughout its length. This statement does not apply if the weight—an external force per unit length—is considered. In this work the direction of gravity is perpendicular to the plane of the loop and the shape of the loop will not depend on the weight of the paper. In addition, for most types of paper the stress is much larger than the weight and gravity can be neglected for any orientation of the loop. (Experimentally the shape does not change noticeably if the loop is held upward or downward.)

Two external forces in opposite directions need to be applied to the edges of the paper to hold the loop. The line of action of these forces is taken to be the x axis. Thus, we look for a solution with the stress F_y equal to zero. In this case, Eq. (1) can be written as a function of the single parameter $a = F_x/EI$:

$$\phi''(s) = a \sin \phi(s) \qquad (2)$$

(the prime and double prime indicate first and second derivatives with respect to s, respectively). We write $\phi'' = (d\phi'/d\phi)\phi'$ and use this relation to obtain two well-known results. First, there is a conserved quantity,

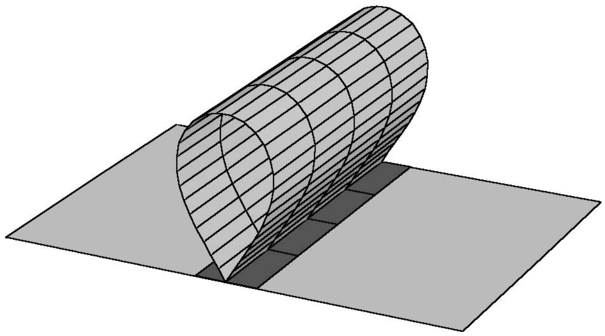

Fig. 1. A sheet of paper folded to form a loop.

$$K=\phi'(s)^2+2a\cos\phi(s), \tag{3}$$

which can be determined from the values of $\phi(s)$ and $\phi'(s)$ at $s=0$, and lets us write:

$$\frac{d\phi}{ds}=\operatorname{sgn}(\phi'(s))\sqrt{K-2a\cos\phi(s)}, \tag{4}$$

where $\operatorname{sgn}(\phi'(s))$ is the sign of $\phi'(s)$. Second, in a domain where $\operatorname{sgn}(\phi'(s))$ is constant, $s(\phi)$ can be determined by integrating Eq. (4). The result in terms of the elliptic integral of the first kind is

$$s(\phi)=s_0+\operatorname{sgn}(\phi')\sqrt{\gamma_a}\left(F\left(\frac{\phi}{2}\,\middle|\,-\gamma_a a\right)-F\left(\frac{\phi(s=0)}{2}\,\middle|\,\gamma_a a\right)\right), \tag{5}$$

where

$$\gamma_a=\frac{4}{K-2a}. \tag{6}$$

The function (5) is valid from $s=0$ to $s_{\max}$ provided that $\operatorname{sgn}(\phi'(s))$ is constant in this domain. If $\phi'(s)$ vanishes at $s=s_1$ between 0 and $s_{\max}$ (there is an inflection point where ϕ' vanishes), the function (5) is valid only from $s=0$ to s_1. In this case Eq. (4) has to be solved again for $s\geqslant s_1$. The new solution will be valid in a domain where $\operatorname{sgn}(\phi'(s))$ is constant, that is, up to another inflection point or $s_{\max}$. The idea is that Eq. (4) has to be solved such that within each piece $\operatorname{sgn}(\phi'(s))$ is constant. Then the solutions are matched at the inflection points.

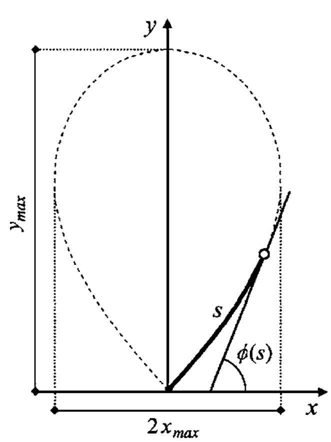

Fig. 2. Cartesian coordinate axes and a loop with aspect ratio $\eta=y_{\max}/(2x_{\max})$.

The Cartesian coordinates of the curve described by Eq. (5) can be obtained by integrating

$$x'(s)=\cos\phi(s), \tag{7a}$$

$$y'(s)=\sin\phi(s), \tag{7b}$$

which follow directly from the definition of the angle ϕ. The ordinate of the curve is a simple function of the angle:

$$y(\phi)=\frac{1}{a}\left(\sqrt{K-2a\cos\phi}-\phi'(s=0)\right), \tag{8}$$

but $x(\phi)$ is given as an elliptic integral. Solving the equations by quadratures does not provide much advantage if we want to avoid knowing E, I, $\phi(s=0)$, $\phi'(s=0)$, and the stress.

To explain the shape of a paper folded into a loop by Eq. (2), we will take a and the value of $\phi'(s=0)$ as the two unknowns. Equations (2) and (7) will be solved numerically for selected values of $\phi(s=0)$ in the range $(0,\pi/2)$ with the boundary conditions:

$$\phi(0)=\phi_0;\quad \phi'(0)=\phi_0';\quad x(0)=0;\quad y(0)=0. \tag{9}$$

The values of a and ϕ_0' for each given ϕ_0 are fixed by the condition that the two edges of the profile are joined. If we use the length of the profile as the unit of length ($s_{\max}=1$), this condition is satisfied if

$$x(1)=0,\quad y(1)=0. \tag{10}$$

Note that the profile begins and ends at the origin.

In Sec. III I will explain how to find a and the value of $\phi'(s=0)$ for each value of $\phi(s=0)$. I will also explain how the aspect ratio $\eta(\phi_0)=y_{\max}/2x_{\max}$ (see Fig. 2) or its inverse $\phi_0(\eta)$ can be determined. Then, after we measure the height and width of the loop of a sheet of paper, we can compute in sequence η, $\phi_0(\eta)$, $a(\phi_0)$, and $\phi_0'(\phi_0)$. The profile can then be calculated and plotted.

III. NUMERICAL ANALYSIS

The first part of the analysis is to determine the parameters a and ϕ_0' as a function of ϕ_0 in the range $(0,\pi/2)$. In the process $y_{\max}$ and $2x_{\max}$ and their ratio η can be computed as a function of ϕ_0. This step is completed after we obtain $\phi_0(\eta)$, $a(\phi_0)$, and $\phi_0'(\phi_0)$. We then compare the loop of our sheet of paper with our numerical results.

The algorithm for determining $\phi_0(\eta)$, $a(\phi_0)$, and $\phi_0'(\phi_0)$ is as follows.

(1) Use Eqs. (2) and (7) and the boundary conditions (9) to define the functions $\xi(\phi_0;a,\phi_0')$ and $\psi(\phi_0;a,\phi_0')$ which return $x(1)$ and $y(1)$ for the input parameters ϕ_0, a, and ϕ_0'. In MATHEMATICA[6] we can use the NDSolve function, which finds a numerical solution to ordinary differential equations.

For $\phi_0=\pi/2$ to $\pi/180$ in steps of one degree do steps (2) and (3).

(2) Solve the equations $\xi(\phi_0;a,\phi_0')=0$ and $\psi(\phi_0;a,\phi_0')=0$ for a and ϕ_0' (I used the FindRoot function in MATHEMATICA). These two equations are equivalent to Eq. (10). Initial estimates of a and ϕ_0' must be given to solve $\xi(\phi_0;a,\phi_0')=0$ and $\psi(\phi_0;a,\phi_0')=0$ numerically. For ϕ_0

$=\pi/2$, I solved Eqs. (2) and (7) with the boundary conditions (9) and plotted the parametric curve $\{x(s),y(s)\}$ for a trial set of values of a and ϕ_0'. The values that gave the end point $\{x(1),y(1)\}$ closest to the origin were used as initial values to solve $\xi(\pi/2;a,\phi_0')=0$ and $\psi(\pi/2;a,\phi_0')=0$. For $\phi_0<\pi/2$, the values of a and ϕ_0' of the previous ϕ_0 can be used as the initial estimates.

(3) Use the values found for a and ϕ_0' to solve Eqs. (2) and (7) with the boundary conditions (9) and compute η. Save η, a, ϕ_0, and ϕ_0'.

The values of the aspect ratio are in the range $(1,\eta_{max}=2.078)$. The aspect ratio is largest for $\phi_0=\pi/2$ and goes to 1 when ϕ_0 goes to zero. The shape of the loop for $\phi_0<0$ takes the form of a cardioid [see Fig. 3(a)] and the aspect ratio decreases from 1 to 0.41 for $\phi_0=-\pi/2$. The shape of the loop changes from a teardrop-like shape for $\phi_0=\pi/2$, to a circle for $\phi_0=0$, and to a cardioid for $\phi_0=-\pi/2$. Here we focus on the teardrop shape. For the cases with $\phi_0<0$, see Problem 2 in Sec. VI.

(4) Use the data saved in step 3 [η and the corresponding values of ϕ_0, ϕ_0', and a that satisfy the boundary conditions in Eq. (9)] to obtain the followings fits for η in the range $\eta=1$ to $\eta_{max}=2.078$:

$$\phi_0(\eta)\approx -3.771+6.669\eta-4.370\eta^2+1.858\eta^3 -0.4309\eta^4+0.04421\eta^5, \qquad (11)$$

$$a(\phi_0)\approx -0.04167+25.16\phi_0, \qquad (12)$$

$$\phi_0'(\phi_0)\approx 6.381-6.717\phi_0-0.7987\phi_0^2. \qquad (13)$$

The first part of the numerical analysis has now been accomplished; Eqs. (11)–(13) are the output of this part.

The second part is to compare the loop of our sheet of paper with our numerical results. This part has to be repeated for all the loops that we have done. The algorithm for calculating the shape of the loop from the aspect ratio that we measure is as follows:

(1) Input η in the range $(1,\eta_{max}=2.078)$ and determine ϕ_0 from Eq. (11).
(2) Solve $\xi(\phi_0;a,\phi_0')=0$ and $\psi(\phi_0;a,\phi_0')=0$ using $a(\phi_0)$ and $\phi_0'(\phi_0)$ given by Eqs. (12) and (13) as initial estimates of a and ϕ_0'.
(3) Solve Eqs. (2) and (7) for $x(s)$, $y(s)$, and $\phi(s)$ with the boundary conditions (9) with ϕ_0 from step (1) and a and ϕ_0' from step (2).
(4) Plot the parametric curve $\{x(s),y(s)\}$.

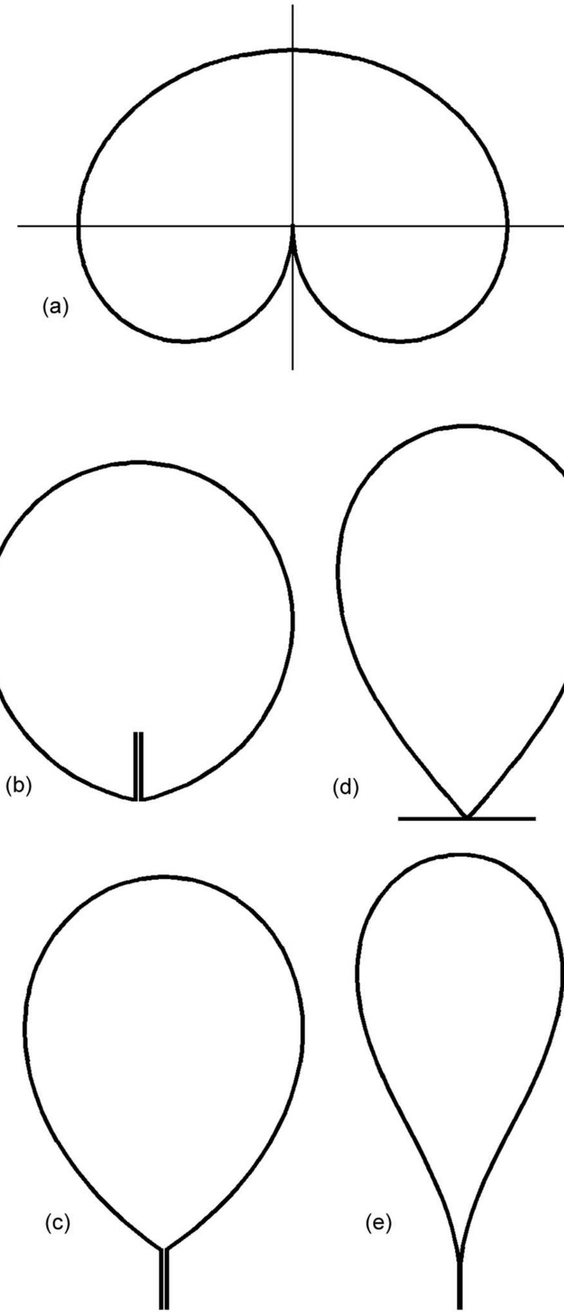

Fig. 3. (a) The edge of a sheet takes the form of a cardioid when opposite edges are joined inward (see Sec. III and Problem 2 in Sec. VI). (b)–(e) Different ways of joining the opposite edges to form an edge with a teardrop profile.

IV. THE TESTS

To form a loop I creased the shorter side of a sheet of paper to form two small wings parallel to the original border (width ~2 cm). Then I joined the wings face to face [Fig. 3(b) or 3(c)], or I placed the paper on a table and joined the internal fold of the wings [Figs. 1 and 3(d)]. The shape of the loop depends on the angle formed by the wings and the plane of the original sheet of paper before creating the loop. I also created loops by joining the edges directly face to face [Fig. 3(e)], or by adhering to a sticky tape 1 cm of the short border of the sheet of paper and the edge of the opposite. Loops with small aspect ratios (~1) are obtained in the case shown in Fig. 3(b). The loop with the largest aspect ratio can be obtained in the case shown in Fig. 3(e).

A qualitative comparison between the experimental and calculated profiles can be done directly on the computer display. Assume that the length of the loop made with the sheet of paper is d (in my case for an A4 sheet, $d=29.7$ cm if sticky tape is used and $d=25.7$ if the edges are folded). We first determine the aspect ratio of the loop with the aid of a ruler and calculate the profile as explained in Sec. III. Then plot on the same graph the loop and a straight line of $\alpha=0.4$ units of length. (I used this value because a loop has unit length and its height is in the range 0.32–0.42.) Select a screen resolution that does not distort shapes (e.g., by drawing a circle and checking that it does not appear as an ellipse). Scale the plot until the length of the straight line on the screen has a length of αd. Then place the loop made with the paper on top of the loop plotted on the computer display. Look through the loop of the paper to compare the agreement between the experimental and calculated profiles.

To obtain more detailed comparisons, I placed the edge of the paper on a scanner. The scanned image is a black background with a well-defined white line corresponding to the edge of the paper. I have also used a digital camera to obtain the profiles of the loops. I placed the paper vertically with the edge on a dark cardboard on the ground. Due to perspective, the image showed two loops corresponding to the two edges of the sheet. Because I did not use any special illumination, the lower edge touching the cardboard usually leads to a profile with higher contrast than the upper edge in the picture.

When the images of the loops are superposed on its mirror image, it is apparent that the profiles are usually somewhat asymmetric. The asymmetry can have several sources: the friction of the edge of the paper with the glass or the cardboard can produce the asymmetry when the paper is placed on it (the paper must be placed with care to minimize this effect); unavoidable imperfections when the paper is creased to form the wings or when they are joined; many sheets of paper (especially those already used) have a border with undulations (visible when the paper lies flat on a table) that could affect the profile of the loop.

In Fig. 4 a typical result with low asymmetry is shown. The background of the image has been removed and black and white have been inverted to give the image shown here. The solid lines correspond to the profile acquired with a scanner and to its mirror image. The dotted line corresponds to the profile calculated using the aspect ratio of the loop as the input. The slight asymmetry of the measured profile is small compared to the good agreement between the numerically calculated profile and that of the actual folded sheet of paper.

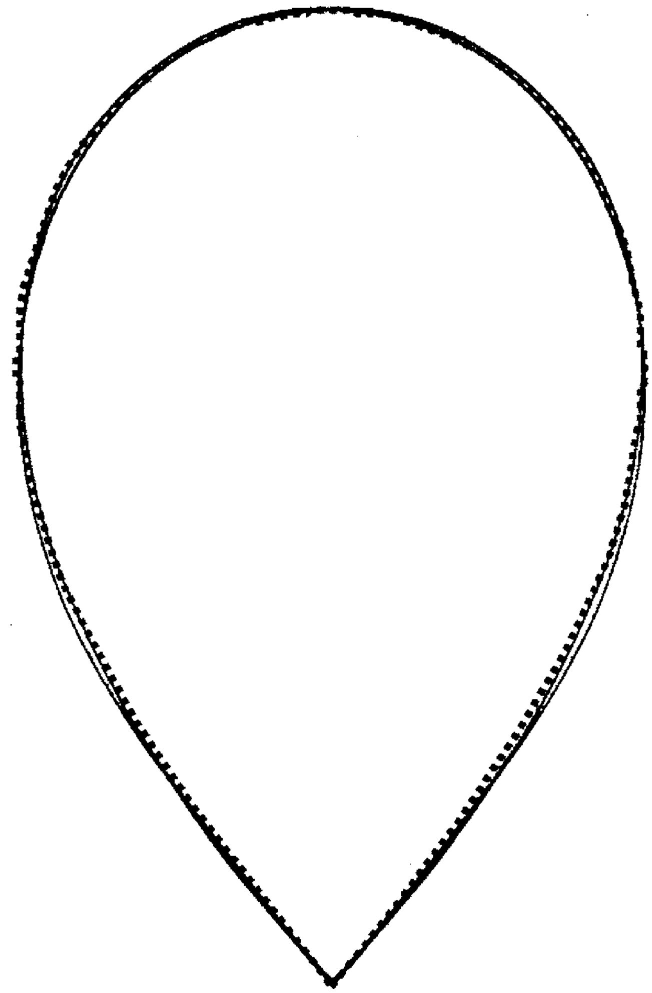

Fig. 4. Comparison between the measured profile (solid line) and the calculated profile (dotted line). There are two solid lines because the profile and its mirror image have been plotted.

V. DISCUSSION

The loop with the largest aspect ratio ($\eta_{\max}=2.078$) has $\phi_0(0)=\pi/2$ and a profile with two symmetric inflection points where ϕ_0' vanishes (the sign of the curvature of the profile changes at these inflection points). The inflection points of the profile approach the origin when η decreases and disappear for $\eta_c=1.524$ where $\phi_0(0)=49.3°$. For $\eta \leq \eta_c$, $\phi'(0)$ is always positive. The profiles for some values of η and $\phi_0(\eta)$ are plotted in Fig. 5. In the limit $\phi(0)\to 0$, the aspect ratio goes to 1 and $\phi'(s)$ goes to $1/2\pi$ (the profile becomes a circle).

For $\eta \leq \eta_c$ the analytical solution (5) with $\mathrm{sgn}(\phi_0')=+1$ can be used directly to obtain $s(\phi)$ ($\phi_0 \leq \phi \leq 2\pi-\phi_0$). For $\eta > \eta_c$ the analytical solution must be determined in three pieces. This complication arises because the analytical solution for $s(\phi)$ is not obtained directly from the differential equation (2), but instead is obtained from the first integral of Eq. (3), where ϕ' appears squared. As a consequence, the exact form of the analytical solution (5) depends on the sign of $\phi'(s)$. For $\eta \leq \eta_c$ $\phi'(s)$ is negative from the origin ($s=0$) to the inflection point; it is positive from this point to the other inflection point and is again negative from this point to $s=1$. If the angle ϕ at the first inflection point is ϕ_i, $s(\phi)$ should be constructed by applying Eq. (5) three times with the appropriate sign between the limits: $\phi_0 \to \phi_i \to 2\pi-\phi_i \to 2\pi-\phi_0$. Another difficulty is that ϕ_i is not known and needs to be calculated numerically.

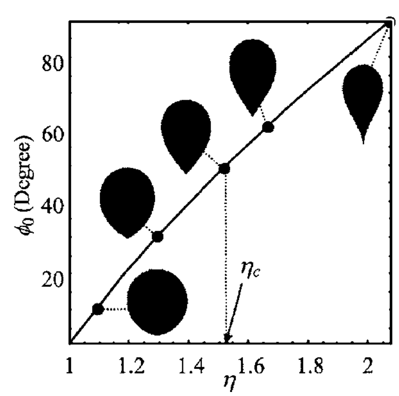

Fig. 5. The value of ϕ_0 as a function of the aspect ratio of the loop. Some loops are shown ($\eta_c=1.524$).

As explained after Eq. (1), the weight of the paper can be neglected. If the weight were important and acted along the y axis (see Fig. 2), Eq. (1) would still hold, but the stress F_y would become a function of s and F_y' would equal the weight per unit length along s. The equations of the elastica (1) and (7) have been applied to different elastic materials taking the weight into account.[3,4,7]

I will mention three applications that show how the equations of the elastica help to determine a physical parameter or understand the shape of a folded material. The equations of the elastica for an initially horizontal lamina with one end clamped were solved numerically to check the classic Peirce's cantilever test for the bending rigidity of textiles.[4] (This problem with a perpendicular constant force applied at the end of the cantilever and no gravity is one of the classic problems with analytical solution.[8]) The equations of the elastica also have been solved to understand the shape of a long strip of paper fed uniformly (and slowly) from a horizontal spool toward a horizontal solid plane where the friction between the paper and the plane is large. It is observed that the sheet folds on itself in a regular manner.[7] A simple fold looks like the loop of Fig. 3(e) if it was rotated 90° and placed on a table. The condition that the paper is fed slowly lets us neglect the inertia of the paper and apply the equations of the elastica. Finally, a brief communication about nanotechnology explained how to form a nanotube with a thin layer on a substrate.[2] By etching, the layer is wrapped up, folded back, and brought into adhesive contact with itself creating a nanotube (the diameter can be as small as 50 nm). In fact, in Ref. 3 the equations of the elastica are solved numerically to obtain the profile of a section of a nanotube. As expected, the cross section of a nanotube for a broad range of the values of their parameters is similar to the fold found in Ref. 7.

As mentioned in Sec. I, the profile of the edge is qualitatively similar to the profile of a teardrop. A typical teardrop hangs from the end of a small tube. There is a technique used to determine the surface tension of a liquid called the *pendant drop method*. It is based on the comparison of the observed shape of the drop to the calculated one, and commercial equipment is available to perform the measurements.[9] The profile of an axisymmetrical drop can be calculated by solving three first-order equations. Two equations arise from the relation between the Cartesian coordinates and the definition of a curve in terms of arclength and slope (see Fig. 2). Therefore, these two equations are equal to Eqs. (7a) and (7b). Any smooth curve in the plane will satisfy these equations. The third equation represents the physical model that produces the curve. For the drop this equation can be written in the form[9]

$$\phi'(s) = b - y(s) - (\sin \phi(s))/x(s), \tag{14}$$

where the parameter b depends on the surface tension, the gravitational field g, the difference between the densities of the drop and the fluid surrounding it, and the curvature of the drop at the apex located at $s=0$, where $\phi(0)=0$. (In our coordinate system the drop will have its apex at the origin, while the paper had the apex at $y_{\max}$.) A comparison of the observed and calculated profile is used to fix the free parameter b and solve for the surface tension. The physical laws involved in drop formation and the final form of the equations, Eq. (14) in contrast to Eq. (2), show that the resemblance of the shape of the profile of folded paper and the shape of a teardrop is superficial. We would reach the same conclusion if we compare the shape of the paper with the edge of a drop sliding down an incline.[10]

VI. SUGGESTED PROBLEMS

Problem 1. In Sec. II we discussed that Euler found the differential equation of the elastica by minimizing the integral of the square of the curvature κ taken along the rod, which worked because such an integral multiplied by $EI/2$ is the elastic energy of the rod.[1,5] For a given sheet of paper, the stored elastic energy when it joins two opposite edges will depend on the shape of its profile. (a) Use the expression for the curvature of a curve $(x(\phi), y(\phi))$ in parametric form to show that $\kappa = d\phi/ds = \phi'$. (b) Write a program to calculate numerically the integral of $\kappa^2 = (\phi'(s))^2$ from $s=0$ to $s=L=1$ for several profiles and plot the results as a function of the aspect ratio η. Check your program for the exact result that for $\eta=1$, the integral is $4\pi^2$. Why? (c) What is the result of the integral for $\eta=\eta_c=1.524$? Has the elastic energy as a function of η some special property at η_c?

Problem 2. If the edges of a paper are joined inward, the edge takes the form of a cardioid [see Fig. 3(a)]. In Sec. III I stated that these cases correspond to $\phi_0 < 0$. The aspect ratio of the loop decreases from 1 for $\phi_0=0$ to 0.41 for $\phi_0=-\pi/2$. (a) Following the first part described in Sec. III, determine the parameters a and ϕ_0' from $\phi_0=-\pi/2$ to $-\pi/180$ and obtain fits for $\phi_0(\eta)$, $a(\phi_0)$, and $\phi_0'(\phi_0)$. (Hint: For $\phi_0=-\pi/2$, the values of a and ϕ_0' are close to -35 and 10, respectively; use these values to set up your initial guess.) (b) Plot the profile of the curve obtained with $\phi_0=-\pi/2$ and compare it with paper folded in the same way. (First, fold the paper to get a C-like profile; second, press the edges inward to obtain a B-like profile; and, finally, hold the edges joined together by pressing them with your fingers.)

Solutions: (1c) 28.11; It is a minimum. (2a) $a(-\pi/2)=-35.242$; $\phi_0'(-\pi/2)=9.829$; $a(\phi_0)\approx 0.3564+27.48\phi_0+2.886\phi_0^2$, $\phi_0'(\phi_0)\approx 6.250-6.309\phi_0-2.568\phi_0^2$.

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