Physics to Mathematics: from Lintearia to Lemniscate – I

R Sridharan

The *elastic curve*, also called *elastica*, is the name given to the shape assumed by a uniform elastic rod when bent into a plane curve under a stress of a certain kind. This curve, defined by James Bernoulli in the 18th century has been an interesting object of study for a long time. The classical work of Tait and Thompson discusses this curve ([1], pp.609-612). This curve has also been called 'Lintearia', being the cross section of a horizontal flexible water-tight cylinder when filled with water, the free surface of which lies in the line of thrust. We assume this to be the y axis. If t denotes the constant circumferential tension at point P at depth x, (*Figure* 1) and ρ, the radius of curvature, we have, $t/\rho = wx$, the pressure of water, so that, $\rho x = t/w = c^2$ ([2], p.89).

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The above physical explanation (see for example [2], pp.87-89), shows the significance of the elastic curve as an object of study in physics. This problem in statics is equivalent to the kinetic problem of the common pendulum. The general equivalence between static and kinetic problems which was established by Kirchoff has since become for the physicists, only a matter of historical interest.

However, for the mathematicians, the elastic curve played a crucial role; it gave birth to the beautiful and profound theory of elliptic integrals and elliptic functions.

To the fragrant memory of a vanished dream.

We begin by deriving the equation of the curve in an integral form. We recall that the radius of curvature ρ of a curve has, with respect to Cartesian co-ordinates, the following formula:

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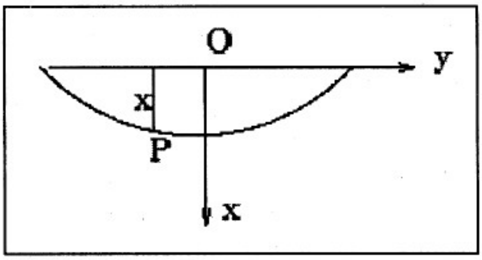
Lemniscate, lintearia, elliptic integrals, Euler's addition theorem.

Figure 1.



$$\rho = \frac{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}.$$

Setting $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, we get, for the elastic curve, the equation

$$\frac{\mathrm{d}p}{(1+p^2)^{3/2}} = \frac{\mathrm{d}x}{\rho} = \frac{x\mathrm{d}x}{c^2}.$$

Substituting $p = \tan\psi$, we get

$$\frac{\sec^2\psi\mathrm{d}\psi}{\sec^3\psi} = \cos\psi\mathrm{d}\psi = \frac{x\mathrm{d}x}{c^2},$$

so that, by integration, we have

$$\sin\psi = \frac{x^2}{2c^2},$$

if one chooses the initial conditions suitably. We therefore have

$$\frac{p}{\sqrt{1+p^2}} = \frac{x^2}{2c^2},$$

so that

$$\frac{p^2}{1+p^2} = \frac{x^4}{4c^4}$$

and hence

$$p^2 = \frac{x^4}{4c^4 - x^4}.$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{\sqrt{a^4 - x^4}},$$

where we set $a^4 = 4c^4$. We finally have

$$y = \int_0^x \frac{x^2 \mathrm{d}x}{\sqrt{a^4 - x^4}},$$

which is the integral equation of the elastic curve.

James Bernoulli knew that such integrals could not be evaluated by the functions known at that time, namely in terms of rational, trigonometric or logarithmic functions. He computed the arc length of the curve and noticed that

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x = \frac{a^2}{\sqrt{a^4 - x^4}} \mathrm{d}x.$$

James Bernoulli knew that such integrals could not be evaluated by the functions known at that time, namely in terms of rational, trigonometric or logarithmic functions.

We shall normalise by putting $a = 1$ from now on, so that the arc length of the elastic curve is given by

$$s = \int_0^x \frac{\mathrm{d}x}{\sqrt{1 - x^4}}.$$

This integral is called the 'Lemniscate integral' for the following reason.

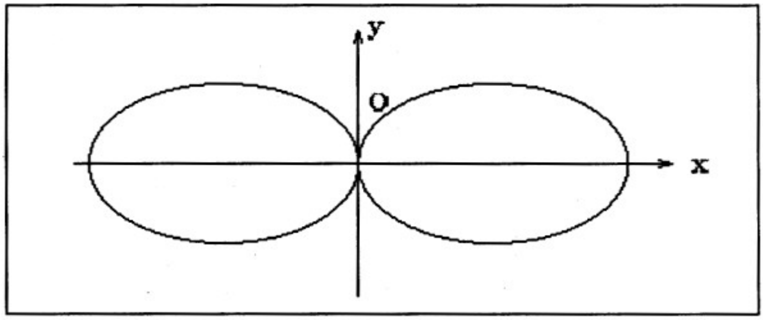
Bernoulli, towards the end of the 17th century defined a remarkable algebraic curve which is called the *Lemniscate*. In fact this curve is one of a family of curves defined as the locus of points in the plane such that the product of its distances from two fixed points is a constant. Taking the two points whose Cartesian coordinates are $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, 0\right)$ and the constant to be $\frac{1}{2}$, we get

$$\left(\left(x + \frac{1}{\sqrt{2}}\right)^2 + y^2\right)\left(\left(x - \frac{1}{\sqrt{2}}\right)^2 + y^2\right) = \frac{1}{4},$$

which yields after some simplification, the equation of the lemniscate as

$$(x^2 + y^2)^2 = x^2 - y^2.$$

Figure 2.



Note that this curve has a parametrisation

$$x^2 = \frac{r^2 + r^4}{2}, \quad y^2 = \frac{r^2 - r^4}{2}.$$

Using this, or the polar equation $r^2 = \cos 2\theta$, the element of arc length $\mathrm{d}s$ is given by $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 = \frac{\mathrm{d}r^2}{1-r^4}$. Thus we see that the arc length of the elastic curve is the same as the arc length of the lemniscate curve. Unlike the elastic curve, the lemniscate curve is given by a polynomial equation; i.e. an algebraic curve. Bernoulli called it the lemniscate curve, as it looks like a bow-tie, or a ribbon folded into a knot. It also resembles a reclining figure of eight (*Figure* 2).

This curve has been a favourite of many mathematicians, in particular of James Bernoulli and his brother John. The Italian mathematician Fagnano studied this curve in the first quarter of the 18th century. What Fagnano did was something beautiful. In fact, the integral

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$$\int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^4}}$$

looks 'very similar' to the integral $\int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^2}}$. This latter integral can be evaluated to be $\sin^{-1} x$ (substitute $x = \sin\theta$. Then we get the integral to be θ; i.e. it gives the arc length of the circle!). If we substitute $x = \frac{2t}{1+t^2}$, we have

$$\int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 2\int_0^t \frac{\mathrm{d}t}{1+t^2},$$

so that the integrand is rationalised. In the lemniscate integral, we have $\sqrt{1-x^4}$ instead of $\sqrt{1-x^2}$. Fagnano must naturally have tried to rationalise $\int \frac{\mathrm{d}x}{\sqrt{1-x^4}}$ by using the analogous substitution

$$x^2 = \frac{2t^2}{1+t^4}.$$

We then get

$$x\mathrm{d}x = \frac{2t(1-t^4)}{(1+t^4)^2}\mathrm{d}t$$

and

$$\sqrt{1-x^4} = \sqrt{1-\frac{4t^4}{(1+t^4)^2}} = \frac{(1-t^4)}{(1+t^4)},$$

so that

$$\frac{\mathrm{d}x}{\sqrt{1-x^4}} = \frac{\sqrt{2}\mathrm{d}t}{\sqrt{1+t^4}}.$$

This must have been a bit of a disappointment, since, one has ended up with a root once again and this substitution did not rationalise the integral. But then by making a new substitution

$$t^2 = \frac{2u^2}{1-u^4},$$

we get

$$t\mathrm{d}t = \frac{2u(1+u^4)}{(1-u^4)^2}$$

and

$$1+t^4 = 1+\frac{4u^4}{(1-u^4)^2} = \frac{(1+u^4)^2}{(1-u^4)^2},$$

so that

$$\frac{\sqrt{2}\mathrm{d}t}{\sqrt{1+t^4}} = \frac{\mathrm{d}u}{\sqrt{1-u^4}}.$$

Fagnano must have realised that he had achieved something remarkable in the process, namely that he had produced twice the arc length of the lemniscate; viz.

$$\int_0^r \frac{\mathrm{d}r}{\sqrt{1-r^4}} = \int_0^u \frac{\mathrm{d}u}{\sqrt{1-u^4}},$$

where

$$r^2 = \frac{2t^2}{1+t^4} = \frac{4u^2(1-u^4)}{(1+u^4)^2}. \tag{1}$$

In fact, what Fagnano achieved was the doubling of a lemniscate arc by ruler and compass!

Indeed, if (ζ, η) are the coordinates of a point of the lemniscate in the first quadrant, then $2\zeta^2 = u^2 + u^4$, $2\eta^2 = u^2 - u^4$. Hence in view of the above equation (1), r^2 is a rational function of $u^2 = \zeta^2 + \eta^2$. Given the point (ζ, η) one can determine a point (x, y) whose coordinates are given by the equations $2x^2 = r^2 + r^4$, $2y^2 = r^2 - r^4$. We note that x and y can be obtained by taking the square roots of rational functions of $\zeta^2 + \eta^2$. Thus the point (x, y) can be obtained by a ruler and compass construction. (We restrict ourselves to the curve in the first quadrant and the arc is chosen so small that its double also lies in the first quadrant). In fact one can check by this method that halving of a lemniscate arc can also be achieved by ruler and compass.

The deeper significance of the doubling of a lemniscate arc by Fagnano was not appreciated by his contemporaries . In fact, from 1718, when Fagnano achieved his doubling, till 1750, nothing happened, until Fagnano's work was sent to the Berlin Academy of which Euler was a prominent member. Very precisely, it was on 23 December 1751 that Euler received Fagnano's work from the Academy. Jacobi, writing years later, says that this should really be fixed as the birth date of the theory of elliptic functions. Euler caught fire and started his remarkable research, first on lemniscate integrals and then more generally on 'elliptic integrals'.

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In order to understand Euler's remarkable contributions, we pause and look at something familiar to all of us. If we take the integral

$$(\sin^{-1} r =) \int_0^r \frac{dr}{\sqrt{1-r^2}},$$

and make the substitution

$$r = 2t\sqrt{1-t^2},$$

we get

$$\int_0^r \frac{dr}{\sqrt{1-r^2}} = 2\int_0^t \frac{dt}{\sqrt{1-t^2}}.$$

This equation really means

$$\sin^{-1} r = 2\sin^{-1} t,$$

where

$$r = \sin(\sin^{-1} r) = \sin(2\sin^{-1} t) =$$
$$2\sin(\sin^{-1} t)\cos(\sin^{-1} t) = 2t\sqrt{1-t^2}.$$

In other words, one has here reformulated the well known formula:

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

However, this is a special case of the more general 'addition' formula, namely

$$\sin(x+y) = \sin x\cos y + \cos x\sin y.$$

If we put $u = \sin x$, $v = \cos x$, we get

$$\sin(\sin^{-1} u + \sin^{-1} v) = u\sqrt{1-v^2} + v\sqrt{1-u^2},$$

or equivalently

$$\sin^{-1} u + \sin^{-1} v = \sin^{-1}(u\sqrt{1-v^2} + v\sqrt{1-u^2}).$$

One has therefore

$$\int_0^u \frac{du}{\sqrt{1-u^2}} + \int_0^v \frac{dv}{\sqrt{1-v^2}} = \int_0^r \frac{dr}{\sqrt{1-r^2}},$$

Very precisely, it was on 23 December 1751 that Euler received Fagnano's work from the Academy. Jacobi, writing years later, says that this should really be fixed as the birth date of the theory of elliptic functions.

Euler's result includes the fact that if two lemniscate arcs are given by their end points, they can be added with the aid of ruler and compass.

where $r = u\sqrt{1-v^2} + v\sqrt{1-u^2}$. The above may be thought of as an 'addition theorem' for the integral $\int_0^x \frac{dx}{\sqrt{1-x^2}}$.

(In view of the difficulties involving the multivalued nature of the inverse trigonometric functions, let us assume that u, v are sufficiently small and the square roots are always positive square roots.)

The fundamental question that Euler asked himself was: Is there a corresponding addition theorem for the lemniscate integral? Of course, he had for his guidance the formula of Fagnano for doubling of the lemniscate integral, namely the substitution

$$r = \frac{2u\sqrt{1-u^4}}{1+u^4}.$$

Note that one makes the substitution $r = 2u\sqrt{1-u^2}$ for doubling in the case of $\frac{dr}{\sqrt{1-r^2}}$, while one has to take $r = u\sqrt{1-v^2} + v\sqrt{1-u^2}$ for the addition formula. Imitating this, Euler took

$$r = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2} \tag{2}$$

and he proved the remarkable result

$$\int_0^u \frac{du}{\sqrt{1-u^4}} + \int_0^v \frac{dv}{\sqrt{1-v^4}} = \int_0^r \frac{dr}{\sqrt{1-r^4}}.$$

In fact, to prove the above, we treat ([3], pp.7-10) r as a constant, and u, v as variables. Note that if $u = 0$, it follows from (2) that $v = r$. Differentiating (2), we get

$$0 = (1+u^2v^2)\times$$

$$\left\{du\sqrt{1-v^4} - \frac{2uv^3}{\sqrt{1-v^4}}dv + dv\sqrt{1-u^4} - \frac{2vu^3}{\sqrt{1-u^4}}du\right\}$$

$$-2(u\sqrt{1-v^4} + v\sqrt{1-u^4})(u^2v\,dv + uv^2du).$$

Let us collect the coefficients du and dv. In fact, a computation gives

$$0=\frac{\mathrm{d}u}{\sqrt{1-u^4}}\{\sqrt{1-u^4}\sqrt{1-v^4}(1-u^2v^2)-2vu(u^2+v^2)\}$$
$$+\frac{\mathrm{d}v}{\sqrt{1-v^4}}\{\sqrt{1-u^4}\sqrt{1-v^4}(1-u^2v^2)-2vu(u^2+v^2)\}.$$

The coefficients of $\frac{\mathrm{d}u}{\sqrt{1-u^4}}$ and $\frac{\mathrm{d}v}{\sqrt{1-v^4}}$ are the *same* and since for $u=0$, this coefficient is $\sqrt{1-r^4}$, for r and u sufficiently near 0, this term remains non-zero so that we can cancel it and get the differential equation of Euler:

$$\frac{\mathrm{d}u}{\sqrt{1-u^4}}+\frac{\mathrm{d}v}{\sqrt{1-v^4}}=0.$$

We note that v is implicitly a function of u. Integrating the above with respect to u with the lower limit 0 for u, we get

$$\int_0^u \frac{\mathrm{d}u}{\sqrt{1-u^4}}+\int_r^v \frac{\mathrm{d}v}{\sqrt{1-v^4}}=0,$$

so that

$$\int_0^u \frac{\mathrm{d}u}{\sqrt{1-u^4}}+\int_0^v \frac{\mathrm{d}v}{\sqrt{1-v^4}}=\int_0^r \frac{\mathrm{d}v}{\sqrt{1-v^4}}.$$

This is the 'addition theorem' of Euler which, for $u=v$ reduces to the theorem of Fagnano on the doubling of the lemniscate arc. Euler's result includes the fact that if two lemniscate arcs are given by their end points, they can be added with the aid of ruler and compass.

Euler began his researches on elliptic integrals (beginning with the lemniscate integral above) with this addition theorem. In the second part, we continue with the interpretation of the addition theorem and with further work of Gauss on the lemniscate.

Suggested Reading

[1] P G Tait and W Thomson, *A Treatise on Natural Philosophy*, C.U.P, 1873.

[2] A G Greenhill, *The applications of elliptic functions*, Reprinted by Dover Publications, New York, 1959.

[3] C L Siegel, *Topics in complex function theory*, Wiley Classics Library, John Wiley, Vol. I, 1969.

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